Derivatives of likelihood ratios and smoothed perturbation analysis for the routing problem

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DERIVEES DE RAPPORTS DE VRAISEMBLANCE ET ANALYSE DE PERTURBATIONS POUR LE PROBLEME DU ROUTAGE

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Résumé

Dans ce rapport nous donnons 3 types de formules permettant de calculer sur une seule trajectoire un estimateur sans biais des dérivées d'un indicateur de performance (de système dynamique à événements discrets) relativement à un paramètre de routage : les formules du type LRM ("likelihood ratio method"), les formules du type RPA ("rare perturbation analysis") de Gong [10], et enfin les formules du type RPA de Brémaud et Vasquez Abad [3]. Ce rapport étend les travaux précédents dans 3 directions : on considère un horizon aléatoire, on obtient les formules dans le cas stationnaire, et l'ordre de dérivation est quelconque. Enfin on montre comment appliquer ces formules à un problème d'optimisation adaptative d'un paramètre de routage.

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Abstract

In this article we present estimates for the gradients of the cycle variables with respect to a thinning parameter in the arrival process of G/G/1 queueing systems. Our estimates belong to the category of the likelihood ratio method (LRM) and smoothed perturbation analysis (SPA) estimates. In particular, we obtain consistent estimates of derivatives of any order in the random or infinite horizon case (stationary and ergodic estimates). We also propose a novel approach of applying the obtained estimates to the adaptive routing design.

1 Introduction

Before we describe the contents of the present article, we shall briefly review the main streams of research in Perturbation Analysis. The goal of Perturbation Analysis (PA) is to obtain unbiased estimates with small variance for the derivative

$$\frac{d}{d\theta}E_\theta[\psi_\theta]$$

of a performance index

$$J(\theta) = E_\theta[\psi_\theta] \tag{1.1}$$

with respect to real parameter θ (we shall not consider the vector case in this article because it does not require any change in the theory that we are going to develop, expect for notation). In finite perturbation analysis, one runs two sets of simulations, for the values θ and $\theta + \Delta\theta$ of the parameter, and evaluate the finite difference derivative (FDD)

$$\frac{J(\theta + \Delta\theta) - J(\theta)}{\Delta\theta}$$

In the case where the probability P_θ is independent of θ, i.e. $J(\theta) = E[\psi_\theta]$, the FDD is

$$E\left[\frac{\psi_{\theta+\Delta\theta} - \psi_\theta}{\Delta\theta}\right]$$

The general case where P_θ depends on θ can very often be reduced to this case. A popular technique, among others, is to generate a random variable X with a θ-dependent c.d.p. F_θ under the form

$$X = X_\theta = F_\theta^{-1}(\theta)$$

where U is a random variable uniformly distributed on [0,1]. Thus for any nonnegative measurable function h_θ

$$E_\theta[h_\theta(X)] = E[h_\theta(F_\theta^{-1}(\theta))] \tag{1.2}$$

This example illustrates one of the basic methods for the reduction to the form $J(\theta) = L[\psi_\theta]$ of a performance index originally in the form (1.1).

In the method of Infinitesimal Perturbation Analysis (IPA) introduced by Ho [12], one expects that the passage to the limit $\Delta\theta \to 0$ is allowed, leading to

$$\frac{dJ(\theta)}{d\theta} = E\left[\lim_{\Delta\theta\to 0} \frac{\psi_{\theta+\Delta\theta} - \psi_\theta}{\Delta\theta}\right]$$

This method, which yields consistent estimates with good variance properties (see, for instance, [13]), depends on the validity of the interchange of limits and expectations in

$$\lim_{\Delta\theta\to 0} E\left[\frac{\psi_{\theta+\Delta\theta} - \psi_{\Delta\theta}}{\Delta\theta}\right] = E\left[\lim_{\Delta\theta\to 0} \frac{\psi_{\theta+\Delta\theta} - \psi_\theta}{\Delta\theta}\right]$$

There are cases where the above equality is true, and others where it is not. In the simple example (1.2) one must at least require that $h_\theta \cdot F_\theta^{-1}$ be continuous and picewise differentiable in θ. In the literature, one has been mostly concerned with the continuity of h_θ [5,6,13].

Although Example (1.2) is relative to a single random variable, the issues are the same for complex discrete event dynamical systems (DEDS): replace X by a random element such as for instance the input sequence (interarrival, and service) into a queueing system and h_θ by a functional determined by the structure of the system. Glasserman [6] gives a clear view of the structural problems of IPA in a somewhat general framework of generalized semi-Markov processes (GSMP) originally introduced by Matthes [18].

The most popular alternative to IPA when continuity issues cannot be overcome is the Score Function Method (SFM) or Likelihood Ratio Method (LRM). These two methods are conceptually the same, the former having been initially devised for random vectors, whereas the latter was applied to functionals of stochastic processes by Glynn [8], Reiman and Weiss [19] and Rubinstein [20]. We shall adopt in this article the acronym LRM.

In the simplest form of LRM, the performance index is of the form

$$J(\theta) = F_\theta[\psi]$$

the dependence on θ being supported by the probability P_θ alone, and the trajectory index $\psi_\theta \equiv \psi$ being independent of the parameter. In LRM, one assumes that for h in a neighborhood of θ, $P_{\theta+h} << P_\theta$ and therefore

$$J(\theta + h) = E_\theta[\psi L(\theta + h)]$$

where $L(\theta + h)$ is the Radon-Nikodym derivative

$$L(\theta + h) = \frac{dP_{\theta+h}}{dP_\theta}$$

Thus

$$\frac{J(\theta+h)-J(\theta)}{h} = E_\theta\left[\psi\frac{L(\theta+h)-L(\theta)}{h}\right]$$

and under favorable circumstances

$$\frac{dJ(\theta)}{d\theta} = E_\theta[\psi\ell(\theta)]$$

where $\ell(\theta)$ is the score function (SF)

$$\ell(\theta) = \frac{dL(\theta)}{d\theta}.$$

Roughly speaking, the domain of application of LRM is broader than that of IPA, however it produces estimates with large variance. More precisely the variance grows linearly with runlength of the simulation. (This phenomena will be explained in subsection 2.2 of the present article; see also [19].)

Another techinque available when IPA does not apply is Smoothed Perturbation Analysis (SPA) introduced by Gong [9]. Here, one uses a smoothing field $\mathcal{G}$ such that

$$\lim_{\Delta\theta} E[E[\frac{\psi_{\theta+\Delta\theta}-\psi_\theta}{\Delta\theta}|\mathcal{G}] = E[\lim_{\Delta\theta} E[\frac{\psi_{\theta+\Delta\theta}-\psi_\theta}{\Delta\theta}|\mathcal{G}]. \tag{1.3}$$

Of course when $\mathcal{G} = \mathcal{F}$, the σ-field containing all the information, SPA reduces to IPA. In some circumanstances, when IPA does not work, a choice of $\mathcal{G} \subset \mathcal{F}$ smoothes out the discontinuities of $\psi(\theta)$ and makes (1.3) valid. The SPA method was introduced by Gong in the routing problem [10]. In Brémaud and Vázquez-Abad [3] the SPA approach was applied to derive formulas similar to the one by Gong [10] with however a difference: whereas Gong's formula can be called "one sided", Brémaud and Vázquez-Abad's formulas are "two-sided". Although clearly the analysis of [3] is SPA, it was called by its authors Rare Perturbation Analysis (RPA) to distinguish it from IPA. Indeed, in IPA one considers for small $\Delta\theta$ the realizations of the P_θ and $P_{\theta+\Delta\theta}$ probabilities which yield close but different trajectories of the DEDS, whereas in RPA, the trajectories are rarely different, but when different, the trajectories are not infinitesimally close as $\Delta\theta > 0$. In [4], Brémaud used another approach, via maximal coupling, which belongs to RPA and yet receives no interpretation as SPA.

In [3] and [4], interesting connections between RPA and LRM have been discovered. More precisely, the gradient formulas of Brémaud and Vázquez-Abad were obtained via a reinterpretation of the LRM formulas, and the maximal coupling approach of Brémaud [4] was also reinterpreted in terms of likelihood ratios. However in the latter reference, the maximal coupling method was presented in view of treating the cases where likelihood ratios were not differentiable, as required by LRM.

We shall now describe the contents of the present article. First of all, we shall be concerned with LRM, SPA and RPA and apply these methods precisely to a case where IPA seems to fail, that is the thinning of a point process where the parameter θ is the probability of discarding a given point of the process. In this respect the present work is a continuation of the article of Gong [10] which it extends and complements in several ways. First of all we treat the random horizon case in order to prepare the way for the stationary case and we give the expressions for the derivatives of arbitrary order. We do the same for the LRM estimates, this adapting the work of Reiman and Weiss [19] for the routing problem. Next we extend the result of Brémaud and Vázquez-Abad [3] to derivative of arbitrary order.

The main aspect of the article is to obtain stationary estimates and/or ergodic estimates. We do so by installing the routing problem in a stationary framework and using the Palm theory of point processes. The random horizon gradient formulas can then be applied and yield the desired estimates.

This approach works without problem because of the domination structure of the routing problem. A similar domination structure was available in [14], who obtained IPA stationary estimates with respect to a parameter governing the law of the services.

Finally, we show how to apply the above theory to the problem of adaptive design of a routing problem between two queues inside a network (in particular the input is not the renewal type) and we give an algorithm of adaptation in the canonical form of the ergodic stochastic approximation problem studied by Métivier and Priouret [17].

In the sequel we shall use the following definition. We shall call ersatz-nth-derivative of ψ_θ any functional $\phi_\theta^{(n)}$ such that $\frac{d^n J(\theta)}{d\theta^n} = E_\theta[\phi_\theta^{(n)}]$.

2 LRM Estimates for Random Stopping Times

2.1 LRM ersatz-derivatives of nth order

In the present subsection we shall derive the LRM estimates for the routing problem. The treatment is very similar to that of Reiman and Weiss [19] who obtained the LRM estimates for derivatives with respect to the rate of a Poisson process. We shall compute the LRM ersatz-nth-derivative with respect to the routing parameter of a functional computed over a random interval under general assumptions similar to those of [19]. In addition, we shall show that, under certain circumstances, the conditions can be considerably relaxed. This is of special interest in the routing problem, where a "domination" structure is available (see Section 2.3). The abstract setting is the following:

Let $\{T_n\}, \{Z_n\}$ and $\{X_n\}$ be three sequences of random elements indexed by $\mathcal{N} = \{0, 1, 2, ...\}$ where (i) $T_n \in \mathcal{R}_+, T_0 \equiv 0$ and $T_n < T_{n+1}$ for all $n \in \mathcal{N}$; The random variables T_n can be thought of as being the arrival times of successive customers into a queueing system. (ii) $Z_n \in E$, some state space endowed with a σ-field $\mathcal{E}$; For instance $Z_n = \sigma_n$ the load (service time) brought into the system by the customer arriving at time T_n. (iii) $X_n \in \{0, 1\}$; X_n can be thought of as a routing variable, taking the value 1 if the customer arriving at time T_n is accepted into the system, and 0 if the customer is rejected.

Define, for all $k \in \mathcal{N}$ the σ-field

$$\mathcal{F}_k = \sigma(Z_0, X_0, T_1, Z_1, X_1, ..., T_k, Z_k, X_k) \tag{2.1.1}$$

Let N be a non-negative $\mathcal{F}_n$-stopping time, that is $N \in \mathcal{N} \cup \{+\infty\}$ and

$$\{N = k\} \in \mathcal{F}_k, \forall k \geq 0 \tag{2.1.2}$$

Let $\{P_\theta, \theta \subset [0,1]\}$ be a family of probability measures defined on the measurable space $(\Omega, \mathcal{F})$ in which live the random variables $T_n, Z_n, X_n (n \in \mathcal{N})$. We shall assume that for all $\theta \subset [0, 1]$

(α) The distribution of $\{T_n, Z_n\}$ does not depend upon θ,

(β) $\{T_n, Z_n\}$ and $\{X_n\}$ are P_θ-independent sequences,

(γ) $\{X_n\}$ is under P_θ an i.i.d sequence with $P(X_n = 1) = \theta$

These hypotheses imply in particular that for all $\theta \in (0,1), \theta + h \in (0,1)$, the probability measures P_θ and $P_{\theta+h}$ are equivalent when restricted to $\mathcal{F}_k$ where k is arbitrary in $\mathcal{N}$, and

$$\frac{dP_{\theta+h}}{dP_\theta}|_{\mathcal{F}_k} = \left(1 + \frac{h}{\theta}\right)^{\sum_{i=0}^k X_i} \left(1 - \frac{h}{1-\theta}\right)^{\sum_{i=0}^k (1-X_i)} \tag{2.1.3}$$

We shall denote the right-hand side of (2.1.3) by $L_k(\theta, h)$.

Let $\mathcal{F}_N$ be the σ-field defined by $\mathcal{F}_N = \{A \in \mathcal{F} | A \cap \{N = k\} \in \mathcal{F}_k, \forall k \geq 0\}$ and let ψ be a functional of $\mathcal{F}_N$, that is a real valued $\mathcal{F}_N$-measurable random variable. In the queueing example developed above, ψ could be N, the number of customers served in a busy cycle, or $\int_0^{T_N} 1_{X(t)=i} dt$, the average time spent at level i of a $\mathcal{N}$-valued process $\{X(t)\}$ such as the queue-length process in a single server queue. We shall assume that for all $\theta \in [0,1]$

$$E_\theta[|\psi|] < \infty \tag{2.1.4}$$

and define the average performance index

$$J(\theta) = E_\theta[\psi]. \tag{2.1.5}$$

Let $\theta \in (0,1)$. We shall introduce two assumptions.

A. $N < \infty$ $P_{\theta'}$-a.s. for all $\theta' \in \mathcal{N}(\theta)$, a neighborhood of θ;

B. $E_\theta[|\psi| e^{\gamma N}] < \infty$ for some $\gamma > 0$.

Define $\bar{X}_i = 1 - X_i$ and

$$M_1 = \sum_{i=0}^N X_i, M_2 = \sum_{i=0}^N \bar{X}_i \tag{2.1.6}$$

The main result of the present subsection is

Theorem 1. Let $\theta \in (0,1)$ and suppose that conditions A and B hold. Then, the nth derivative of $J(\theta)$ at θ exists and is given by the formula

$$\frac{d^n J(\theta)}{d\theta^n} = n! E_\theta \left[\psi \left\{ \sum_{p=0}^n (-1)^p \begin{pmatrix} M_1 \\ n-p \end{pmatrix} \begin{pmatrix} M_2 \\ p \end{pmatrix} \frac{1}{\theta^{n-p}} \frac{1}{(1-\theta)^p} \right\} \right] \tag{2.1.7}$$

with the convention $\begin{pmatrix} m \\ j \end{pmatrix} = 0$ if $j > m$.

We shall need the following

Lemma 1. Let $\theta \in (0,1)$ and assume that condition A holds true. Then for any h such that $\theta + h \in \mathcal{N}(\theta), P_{\theta+h} << P_\theta$ on $\mathcal{F}_N$ and

$$\frac{dP_{\theta+h}}{dP_\theta}|_{\mathcal{F}_N} = L_N(\theta, h) \tag{2.1.8}$$

where $L_k(\theta, h)$ is for each $k \in \mathcal{N}$ the right hand side of (2.1.3).

Proof: One must show that for any $A \in \mathcal{F}_N$

$$E_{\theta+h}[1_A] = E_\theta[1_A L_N(\theta, h)]$$

But since $N < \infty$, $P_{\theta+h}$ a.s.

$$E_{\theta+h}[1_A] = \sum_{n=0}^{\infty} E_{\theta+h}[1_{A\cap\{N=n\}}]$$

and since $A \cap \{N = n\} \in \mathcal{F}_n$ for all $n \geq 0$,

$$E_{\theta+h}[1_A] = \sum_{n=0}^{\infty} E_\theta[1_{A\cap\{N=n\}} L_n(\theta,h)] = E_\theta[1_A L_N(\theta,h)]$$

where we have used the assumption that $N < \infty$, P_θ a.s. to obtain the last equality. ∎

An elementary computation gives

$$\frac{d^n L_N(\theta,h)}{dh^n} = n! \sum_{p=0}^{n} (-1)^p \binom{M_1}{n-p} \frac{1}{\theta^{n-p}} \left(1+\frac{h}{\theta}\right)^{M_1-n+p} \binom{M_2}{p} \frac{1}{(1-\theta)^p} \left(1-\frac{h}{1-\theta}\right)^{M_2-p} \tag{2.1.9}$$

Proof of Theorem 1. Denoting by $L_N^{(j)}(\theta,h)$ the j-th derivative of $L_N(\theta,h)$ w.r. to h, we have by Taylor's formula

$$L_N(\theta,h) = 1 + \sum_{j=1}^{n} \frac{L_N^{(j)}(\theta,0)}{j!} h^j + h^{n+1} R_{n+1}(h) \tag{2.1.10}$$

where

$$R_{n+1}(h) = L_N^{(n+1)}(\theta, \alpha h)$$

for some (random) $\alpha \in (0,1)$. Condition B implies that $E_\theta[\psi N^j] < \infty$ for all j, and in particular $E[\psi L_N^{(j)}(\theta,0)] < \infty$ for all j. It also implies that $E_\theta[L_N^{(n+1)}(\theta,\alpha h)\psi] < K$, K independent of h, in a h-neighborhood of θ. Therefore, multiplying both sides of (2.2.10) by ψ and taking expectations w.r. to P_θ, we obtain

$$E_{\theta+h}[\psi] = E_\theta[\psi] + \sum_{j=1}^{n} \frac{h^j}{j!} E_\theta[\psi L_N^{(j)}(\theta,0)] + h^n O(h)$$

and this gives the announced result. ∎

We shall now use another condition than B which will be particularly useful in the routing problem. The condition is

B'. $E_{\theta+t}[|\psi| N^{n+1}] < M$ independent of t in a t neighborhood of θ.

Theorem 1'. Let $\theta \in (0,1)$ and suppose that condition A and B' hold true. Then, the nth derivative of $J(\theta)$ exists and is given by a formula (2.2.7).

Proof: Using Lagrange's residual

$$L_N(\theta,h) = 1 + \sum_{j=1}^{n} \frac{L_N^{(j)}(\theta,0)}{j!} h^j + \int_0^h (h-t)^n L_N^{(n+1)}(\theta,t)dt$$

we have

$$\left| E_\theta \left[\psi \int_0^h (h-t)^n L_N^{(n+1)}(\theta,t)dt \right] \right|$$

$$= \left| \int_0^h (h-t)^n E_{\theta+t} \left[\psi \left\{ \sum_{p=0}^{n+1} (-1)^p \binom{M_1}{n+1-p} \binom{M_2}{p} \frac{1}{(\theta+h)^{n+1-p}} \frac{1}{(1-\theta-h)^p} \right\} \right] dt \right|$$

and therefore, in view of assumption B', for some $K < \infty$ the above quantity is

$$\leq K \int_0^h (h-t)^n dt = h^n O(h). \tag{2.1.11}$$

Since condition B' also guarantees the existence and finiteness of the quantity $E_\theta[L_N^{(j)}(\theta,0)\psi]$, the theorem is proven. ■

2.2 An alternative form of the LRM estimate

We shall give an alternative expression for the ersatz-derivatives of Theorem 1 and 1′. Similar approach was proposed independently (and in a prior work) by Glasserman [7] who also used martingale calculus as we shall do now. It is interesting to note that this alternative expression is the one that we obtain directly in the stationary case, as we shall see in subsection 2.3.

Define for each $t \in \mathcal{R}_+$ the σ-field $\mathcal{G}_t$ by

$$\mathcal{G}_t = \mathcal{F}_t^N \tag{2.2.1}$$

Because $N(t) < \infty$, P_θ-a.s. for all $\theta \in (0,1)$, the same proof as in Lemma 1 yields $P_{\theta+h} << P_\theta$ on $\mathcal{G}_t$ and

$$\frac{dP_{\theta+h}}{dP_\theta}|_{\mathcal{G}_t} \stackrel{\text{def}}{=} \mathcal{L}_t(\theta,h) = \left(1+\frac{h}{\theta}\right)^{\hat{M}_1(t)}\left(1-\frac{h}{1-\theta}\right)^{\hat{M}_2(t)} \tag{2.2.2}$$

where

$$\hat{M}_1(t) = \sum_{i=0}^{N(t)} X_i, \quad \hat{M}_2(t) = \sum_{i=0}^{N(t)} (1-X_i) \tag{2.2.3}$$

The process $\{\mathcal{L}_t(\theta,h)\}, t \in \mathcal{R}_+$, is a $(P_\theta,\mathcal{G}_t)$-martingale, in particular, for all $0 \leq s \leq t$, all h such that $\theta + h \in (0,1)$

$$E_\theta\left[\frac{\mathcal{L}_t(\theta,h)-1}{h}|\mathcal{F}_s\right] = \frac{\mathcal{L}_s(\theta,h)-1}{h}$$

that is

$$E_\theta\left[1_A\frac{\mathcal{L}_t(\theta,h)-1}{h}\right] = E_\theta\left[1_A\frac{\mathcal{L}_s(\theta,h)-1}{h}\right]$$

for all $A \in \mathcal{G}_s$, all $0 \leq s \leq t$. We can apply Theorem 1′ (since $\mathcal{L}_t(\theta,h) = L_{N(t)}(\theta,h)$), with $\psi = 1_A$, to obtain by passing to the limit $(h - 0)$ in the latter equality:

$$E_\theta\left[1_A\left(\frac{\hat{M}_1(t)}{\theta} - \frac{\hat{M}_2(t)}{1-\theta}\right)\right] = E_\theta\left[1_A\left(\frac{\hat{M}_1(s)}{\theta} - \frac{\hat{M}_2(s)}{1-\theta}\right)\right].$$

The condition B' reads in this case

$$E_{\theta+h}[N(t)^2] < M \text{ independent of } h \text{ in a } h\text{-neighborhood of } 0. \tag{2.2.4}$$

Since the latter equality is true for all $A \in \mathcal{G}_s$, $0 \leq s \leq t$, we have shown that

$$\mathcal{L}_t'(\theta,0) \stackrel{\text{def}}{=} \frac{\hat{M}_1(t)}{\theta} - \frac{\hat{M}_2(t)}{1-\theta} \tag{2.2.5}$$

defines a $(P_\theta, \mathcal{G}_t)$-martingale $\{\mathcal{L}_t'(\theta, 0)\}, t \in \mathcal{R}$. Of course similar results hold for higher derivatives of $\mathcal{L}_t(\theta, h)$ at $h = 0$. Although we could have derived the above result more directly in the present situation, we have prefered to give a proof which can be easily extended to other situations. Let now $\{Z(t)\}, t \in \mathcal{R}$, some real valued nonnegative process adapted to the filtration $\{\mathcal{G}_t\}$, and apply Theorem $1'$ to

$$\psi = \int_0^t Z(s)ds \tag{2.2.6}$$

We shall suppose in addition, but this is not of crucial importance, that $\{Z(t)\}$ is bounded

$$Z(t) \leq K, \forall t. \tag{2.2.7}$$

Therefore, condition B' of Theorem $1'$ is verified as soon as (2.2.7) is true, and then

$$\frac{d}{d\theta} E_\theta \left[\int_0^t Z(s)ds\right] = E_\theta \left[\left(\int_0^t Z(s)ds\right) \mathcal{L}_t'(\theta, 0)\right] \tag{2.2.8}$$

Integration by parts gives

$$\left(\int_0^t Z(s)ds\right) \mathcal{L}_t'(\theta, 0) = \int_0^t Z(s) d\mathcal{L}_s'(\theta, 0) + \int_0^t Z(s)\mathcal{L}_s'(\theta, 0)ds \tag{2.2.9}$$

We shall suppose that $\{Z(t)\}$ is left continuous which is not a restriction at all in practical situations: for instance if $\{Z(t)\}$ were a corlol (continuous on the right limited on the left) process with a countable number of discontinuity points in any finite interval, $Z(s)$ could be replaced by $Z(s^-)$ in the expression (2.2.6) for ψ. Since $\{Z(t)\}$ is, under these various assumptions, a bounded predictable process,

$$m(t, 0) = \int_0^t Z(s) d\mathcal{L}_s'(\theta, 0) \tag{2.2.10}$$

defines a $(P_\theta, \mathcal{G}_t)$-martingale with 0 mean (see for instance [2], p.) and therefore it's P_θ-expected value is 0. Therefore, in view of (2.2.8) and (2.2.9)

$$\frac{d}{d\theta} E_\theta \left[\int_0^t Z(s)ds\right] = E_\theta \left[\int_0^t Z(s)\mathcal{L}_s'(\theta, 0)ds\right]. \tag{2.2.11}$$

Thus, both

$$\left(\int_0^t Z(s)ds\right) \mathcal{L}_t'(\theta, 0) \text{ and } \int_0^t Z(s)\mathcal{L}_s'(\theta, 0)ds \tag{2.2.12}$$

are consistent estimates of

$$\frac{d}{d\theta} E_\theta \left[\int_0^t Z(s)ds\right].$$

The variance of the second one was believed by the authors to be, under general circumstances, smaller than the variance of the first one. The zero mean martingale (2.2.10) can be considered as a "noise" which we suppressed from the original estimate. Glasserman [7] gives conterexample, where the first estimate of (2.2.12) remain better than the second one, and gives conditiones guaranteeing that the second estimate is better than the first one.

Although some improvement may be claimed under certain circumstances when using the second estimate of (2.2.12), both estimates in (2.2.12) are not satisfactory in applications. Indeed, consider, for example the following situation: $\{T_n\}, n \geq 0$, forms a renewal point process with interarrival cumulative distribution function $H(x)$ and $\{\sigma_n\}, n \geq 0$, is an iid sequence of service times, independent of $\{T_n\}$, and with c.d.f. $G(x)$; $Z_n \equiv \sigma_n$. Take for $Z(t)$ a bounded function f of the workload $W(t)$ and of the congestion $X(t)$ at time t (under some discipline, say FIFO)

$$Z(t) = f(W(t), X(t)) \tag{2.2.13}$$

We shall assume that

$$\rho = \lambda E[\sigma] = \lambda \int_0^\infty t dG(t) < 1 \tag{2.2.14}$$

where

$$\lambda^{-1} = \int_0^\infty t dH(t) \tag{2.2.15}$$

This implies in particular that for all $\theta \in [0,1)$, the θ-system is stable, and in particular

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t f(W(s), X(s))ds = \int_{\mathcal{R}_+} \int_{R_+} f(w,z) Q_{\theta,\infty}(dw, dx), P_\theta - a.s. \tag{2.2.16}$$

where $Q_{\theta,\infty}$ is the limit distribution of $(W(t), X(t))$ as $t \to \infty$ (it turns out that we do not need to assume f continuous, but we shall not dwell on this). We shall denote the RHS of (2.2.16)

$$E_\theta[f(W(\infty), X(\infty)] \tag{2.2.17}$$

By dominated convergence

$$E_\theta \left[\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t f(W(s), X(s))ds \right] = \int_{R_+} \int_{R_+} f(w,z) Q_{\theta,\infty}(dw, dx) P_\theta ds \tag{2.2.18}$$

and therefore

$$\frac{d}{d\theta} E_\theta \left[f(W(\infty), X(\infty)) \right] = \lim_{t\uparrow\infty} \frac{1}{t} E_\theta \left[\int_0^t f(W(s), X(s)) \mathcal{L}'_s(\theta, 0) ds \right] \tag{2.2.19}$$

Thus, a nearly consistent estimate of the derivative of (2.2.17) is

$$\frac{1}{t} \int_0^t f(W(s), X(s)) \mathcal{L}'_s(\theta, 0) ds \tag{2.2.20}$$

The larger t is, the closer to consistency we get. However, for large t, the variance of (2.2.20) is bad, as we can see by taking $f \equiv 1$ for example. Using the unique increasing $\mathcal{F}_t$-predictable process $\{< \mathcal{L}'(\theta, 0) >_t\}$ such that

$$(\mathcal{L}'_t(\theta, t))^2 - < \mathcal{L}'(\theta, 0) >_t \tag{2.2.21}$$

is a $(P_\theta, \mathcal{G}_t)$ martingale (see Liptser and Shiryayev [15], for instance; we do not discuss here the precise conditions of validity of the above statement), we have, by integration by parts

$$\int_0^t \mathcal{L}'_s(\theta, 0) ds = t\mathcal{L}'_t(\theta, 0) - \int_0^t s d\mathcal{L}'_s(\theta, 0)$$

and therefore

$$\begin{aligned} E_\theta \left[\left(\frac{1}{t} \int_0^t \mathcal{L}'_s(\theta, 0) ds \right)^2 \right] &= E_\theta[\mathcal{L}'_t(\theta, 0)^2] + \frac{1}{t^2} E_\theta \left[\left(\int_0^t s d\mathcal{L}'_s(\theta, 0) ds \right)^2 \right] \\ &\quad - \frac{2}{t} E_\theta \left[\mathcal{L}'_t(\theta, 0) \int_0^t s d\mathcal{L}'_s(\theta, 0) ds \right] \end{aligned}$$

We shall use formally without precising the conditions of applications one of the basic rule of martingale calculus (see [2], for instance).

$$E_\theta\left[\int_0^t f_s d\mathcal{L}'_s(\theta,0)\cdot\int_0^t g_s d\mathcal{L}'_s(\theta,0)\right] = E_\theta\left[\int_0^t f_s g_s d < \mathcal{L}'(\theta,0) >_s\right]$$

which requires $\{f_t\}$ and $\{g_t\}$ to be $\mathcal{F}_t$-predictable, plus some other integrability conditions. In particular

$$\begin{aligned} E_\theta\left[\left(\int_0^t s d\mathcal{L}'_s(\theta,0)\right)^2\right] &= E_\theta\left[\int_0^t s^2 d < \mathcal{L}'(\theta,0) >_s\right] \\ E_\theta\left[\mathcal{L}'_t(\theta,0)\int_0^t s d\mathcal{L}'_s(\theta,0)\right] &= E_\theta\left[\int_0^t s d < \mathcal{L}'(\theta,0) >_s\right] \\ E_\theta[(\mathcal{L}'_s(\theta,0))^2] &= E_\theta[< \mathcal{L}'(\theta,0) >_t] \end{aligned}$$

Therefore

$$E_\theta\left[\frac{1}{t}\int_0^t \mathcal{L}'_s(\theta,0)ds\right] = \frac{1}{t^2}E_\theta\left[\int_0^t (t-s)^2 d < \mathcal{L}'(\theta,0) >_s\right] \tag{2.2.22}$$

In the example of interest to us, and when $\{T_n\}$ is a Poisson process of rate λ, it can be shown that

$$< \mathcal{L}'(\theta,0) >_t = t\left(\frac{\lambda}{\theta(1-\theta)}\right) \tag{2.2.23}$$

Thus the variance of the LRM estimate grows linearly with t. Of course, this was demonstrated in a particular case of no practical value ($f \equiv 1$!), but similar computations can be made for more realistic situations which lead to the same qualitative result. In the above computations, we have maintained a certain level of abstraction so that it should be clear to the interested reader what to do in more general situations. In particular we have used martingale calculus, which is a natural tool when likelihood ratios intervene in a dynamical structure.

In Section 2.4 we shall show that the "localization" of the likelihood ratio occurs in a natural way in the stationary case. For a performance index $J(\theta)$ of the form (2.2.13), and in the renewal situation above, with the stability assumption (2.2.14), we shall prove that:

$$\frac{d}{d\theta}J(\theta) = \lim_{t\uparrow\infty}\frac{1}{t}\int_0^t f(W(s),X(s))\ell(\theta,s)ds \tag{2.2.24}$$

where

$$\ell(\theta,t) = \frac{M_1(t)}{\theta} - \frac{M_2(t)}{1-\theta} \tag{2.2.25}$$

with

$$M_1(t) = \sum_i X_i 1_{[R_-(\theta,t),t)}(T_i),\ M_2(t) = \sum_i (1-X_i)1_{[R_-(\theta,t),t)}(T_i) \tag{2.2.26}$$

Here $R_-(\theta,t)$ is the last regeneration point of the θ-system before ($\le$) time t, namely it is the last time of arrival before ($\le$) time t of a cutsomer accepted in the θ-system and finding the queue empty. Thus $\ell(\theta,t)$ is "locally" computed over the current cycle at time t. This localization is possible because $f(X(t),W(t))$ can also be computed locally over the current cycle at time t.

We shall prove the above result in a more general situation, where the input process is not renewal, but only stationary in a sense to be precised. This generalization is of fundamental importance if one wishes to consider the routing problem in a network situation, where the queue under consideration recieves its input from another queue in equilibrium. Indeed, the input process generally fails to be renewal in such cases. We shall talk more about the routing problem in a (feed forward) network in Section 4.

3 SPA Estimates for Random Stopping Times

3.1 SPA ersatz-derivatives of n^{th} order

In this subsection we shall derive the SPA estimates for the routing problem. These estimates were obtained by Gong [10] for a fixed time horizon or fixed number of arrivals horizon. We shall extend them to an arbitrary random horizon (with however some moment conditions) and give explicit form of the n^{th} derivative. Our treatment is similar to that of Brémaud and Vázquez-Abad [3] in the Poisson case, where derivatives with respect to the rate of a Poisson process were sought. The formal setting is the following:

There are two sequences $\{T_n\}, \{Z_n\}$ indexed by $n \in \mathcal{N}$, with $0 \equiv T_0 < T_1 < ... < \infty$, and there is a family, indexed by $h \in [0, \theta]$ for some $\theta \in (0, 1)$, of sequences $\{B_n(h)\}$ indexed by $\mathcal{N}$, and $\{0, 1\}$-valued. The random variables Z_n take their values in an arbitrary measurable space $(E, \mathcal{E})$, however, one can think of Z_n being the service terms σ_n required by the customer $\#n$, arriving at time T_n. When $B_n(h) = 1$, the customer $\#n$, who was initially accepted at time T_n in the θ-system, is rejected in the $(\theta - h)$-system. Otherwise, if $B_n(h) = 0$, this customer is still accepted in the $(\theta - h)$-system.

The probability P governing all the $(\theta - h)$-systems at the same time, for $h \in [0, \theta]$, is such that, conditionally with respect to $\{T_n\}$ and $\{Z_n\}$, the sequence $\{B_n(h)\}$ is iid with

$$P(B_n(h) = 1|T, Z) = \frac{h}{\theta} \tag{3.1.1}$$

with the obvious notation: T, Z represent the whole sequences $\{T_n\}$, $\{Z_n\}$.

Let now $N \in \mathcal{N}_+ = \mathcal{N} \cup \{+\infty\}$ be a $\mathcal{G}_k$-stopping time for the history $\{\mathcal{G}_n\}$, $n \subset \mathcal{N}$, defined by

$$\mathcal{G}_k = \sigma(T_0, Z_0, ..., T_k, Z_k) \tag{3.1.2}$$

and let $\{Z(\theta - h)\}, h \in [0, \theta]$, be a family of random variables described in terms of a single functional ψ:

$$Z(\theta - h) = \psi(B_0(h), ..., B_N(h), T_0, ..., T_N, Z_0, ..., Z_N) \tag{3.1.3}$$

For $h = 0$, in view of (3.1.1), we have $B_n(\theta) \equiv 0$, and therefore

$$Z(\theta) = \psi(0, ..., 0, T_0, ..., T_N, Z_0, ..., Z_n) \tag{3.1.4}$$

In order to make the computations more readable, we shall introduce a convenient system of notation. For instance

$$B_0^N(h) = (B_0(h), ..., B_N(h)) \tag{3.1.5}$$

and the like for T_0^N, Z_0^N. Therefore

$$Z(\theta - h) = \psi(B_0^N(h), T_0^N, Z_0^N), Z(\theta) = \psi(0, T_0^n, Z_0^n) \tag{3.1.6}$$

Also, we shall introduce the following random sets for $m \in \mathcal{N}_+$,

$$\mathcal{R}(m, N) = \{(i_1, \dots, i_m) | 0 \le i_1 < i_2 < \dots < i_m \le N\} \tag{3.1.7}$$

Also we shall denote for $(i_1, \dots, i_m) \in \mathcal{R}(m, N)$ by $1_{i_1,\dots,i_m}$ the vector $(b_0, \dots, b_N)$ where $b_j = 1$ if $j \in \{i_1, \dots, i_m\}$, and $b_j = 0$ otherwise. Also we shall define for $(i_1, \dots, i_m) \in \mathcal{R}(m, N)$

$$\psi_{i_1,\dots,i_m} = \psi(1_{i_1,\dots,i_m}, T_0^N, Z_0^N), \tag{3.1.8}$$

and finally we denote

$$\psi_\emptyset = \psi(0, T_0^N, Z_0^N) \tag{3.1.9}$$

In these notations

$$Z(\theta) - Z(\theta - h) = \sum_{m=1}^{\infty} \sum_{(i_1,\dots,i_m)} \left\{(\psi_\emptyset - \psi_{i_1,\dots,i_m}) 1_{A_{i_1,\dots,i_m}} 1_{B_0^N(h) = 1_{i_1,\dots,i_m}}\right\} \tag{3.1.10}$$

where

$$A_{i_1,\dots,i_m} = \{\omega | (i_1, \dots, i_m) \in \mathcal{R}(m, N(\omega))\}. \tag{3.1.11}$$

We shall introduce the following assumption.

C. There exists a non-negative random variable $Y(\theta)$ independent of h, such that

$$|Z(\theta - h)| \le Y(\theta), \ \forall h \in [0, \theta] \tag{3.1.12}$$

and such that for some $n \ge 1$

$$E[Y(\theta)(N + 1)^{n+1}] < \infty \tag{3.1.13}$$

As we shall see, the random variable $Y(\theta)$ presents itself in a natural way in the routing problem. Condition C is of the same nature as condition B' of subsection 2.1. When C is satisfied,

$$J(\theta - h) = E[Z(\theta - h)] \tag{3.1.14}$$

exists for all $h \in [0, \theta]$, and, from (3.1.10)

$$J(\theta - h) - J(\theta) = \sum_{m=1}^{\infty} \sum_{i_1,\dots,i_m} E\left[(\psi_\emptyset - \psi_{i_1,\dots,i_m}) 1_{A_{i_1,\dots,i_m}} 1_{B_0^N(h) = 1_{i_1,\dots,i_m}}\right] \tag{3.1.15}$$

Observing that $(\psi_\emptyset - \psi_{i_1,\dots,i_m}) 1_{A_{i_1,\dots,i_m}}$ is a $\mathcal{G}_N$-measurable random variable, and that $\{B_n(h)\}$ is for each h, a $\{0, 1\}$ valued sequence conditionally iid given $\{Z_n\}$, with the conditional distribution (3.1.1), and using the fact that N is a $\mathcal{G}_k$-stopping time.

$$J(\theta - h) - J(\theta) = \sum_{m=1}^{\infty} \sum_{i_1,\dots,i_m} E\left[(\psi_\emptyset - \psi_{i_1,\dots,i_m}) 1_{A_{i_1,\dots,i_m}} \left(\frac{h}{\theta}\right)^m \left(1 - \frac{h}{\theta}\right)^{N+1-m}\right] \tag{3.1.16}$$

Using the binomial formula to write

$$\left(1 - \frac{h}{\theta}\right)^{N+1-m} = \sum_{k=0}^{N+1-m} (-1)^h \left(\frac{h}{\theta}\right)^k \binom{k}{N + 1 - m} \tag{3.1.17}$$

and collecting the terms in (3.1.16) in factor of $h^j (j \geq 1)$ we obtain

$$J(\theta - h) = J(\theta) + \sum_{z=1}^{\infty} \left(\frac{h}{\theta}\right)^j E\left\{ \sum_{i_1,\ldots,i_m} (\psi_\emptyset - \psi_{i_1,\ldots,i_m}) 1_{A_{i_1,\ldots,i_m}} \binom{k}{N+1-m} \right\} \tag{3.1.18}$$

That is

$$\begin{aligned} J(\theta - h) &= J(\theta) + \sum_{z=1}^{n} h^j \left(\frac{1}{\theta^j}\right) E\left[\sum_{k=0}^{j} (-1)^k \sum_{m=1}^{j-k} \left\{ \sum_{i_1\ldots i_m} (\psi_\emptyset - \psi_{i_1,\ldots,i_m}) 1_{A_{i_1,\ldots i_m}} \binom{k}{N+1-m} \right\} \right] \\ &+ h^{n+1} R_{n+1}(h) \end{aligned} \tag{3.1.19}$$

where

$$R_{n+1}(h) = \frac{1}{\theta^{n+1}} \sum_{l=1}^{\infty} \left(\frac{h}{\theta}\right)^l \sum_{k=0}^{n+k} (-1)^k E\left[\sum_{m=1}^{n+l-k} (\psi_\emptyset - \psi_{i_1,\ldots,i_m}) 1_{A_{i_1,\ldots,i_m}} \binom{k}{N+1-m} \right] \tag{3.1.20}$$

Theorem 2. Under the condition C, (3.1.19) holds with $R_{n+1}(h)$ bounded in $h \in [0, \theta]$. (In other words we have obtained a Taylor expansion of $J(\theta - h)$ for $h > 0$.)
We shall do the complete proof for $n = 1$ to obtain the formula of Gong [10],

$$\frac{dJ(\theta)}{d\theta} = \frac{1}{\theta} E[\sum_{i=0}^{N} (\psi_\emptyset - \psi_i)]. \tag{3.1.21}$$

Proof. When $n = 1$ we have

$$\begin{aligned} J(\theta) - J(\theta - h) &= E\left[\sum_i (\psi_\emptyset - \psi_i) 1_{A_i} \frac{h}{\theta} \left(1 - \frac{h}{\theta}\right)^N \right] \\ &+ E[(Z(\theta) - Z(\theta - h)) 1_{|B_0^N(h)|>1}] \\ &= \frac{h}{\theta} E\left[\sum_{i=0}^{N} (\psi_\emptyset - \psi_i) \right] \\ &+ \frac{h}{\theta} E\left[\sum_{i=1}^{N} (\psi_\emptyset - \psi_i) \left[\left(1 - \frac{h}{\theta}\right)^n - 1 \right] \right] \\ &+ E[(Z(\theta) - Z(\theta - h)) 1_{|B_0^N(h)|>1}] \end{aligned}$$

Therefore

$$\begin{aligned} |J(\theta) - J(\theta - h) - \frac{h}{\theta} E\left[\sum_{i=1}^{N} (\psi_\emptyset - \psi_i) \right]| &\leq \frac{h}{\theta} E\left[NY(\theta) \left[\left(1 - \frac{h}{\theta}\right)^N - 1 \right] \right] + \\ & 2E\left[Y(\theta) \left\{ 1 - \left(1 - \frac{h}{\theta}\right)^{N+1} - (N+1)\frac{h}{\theta} \left(1 - \frac{h}{\theta}\right)^N \right\} \right]. \end{aligned}$$

The first term after the inequality sign is bounded in absolute value by $\frac{h}{\theta} E[NY(\theta)]$. The last bracketed quantity $\{\}$ is, by Taylor's formula equal to

$$\frac{h^2}{\theta^2} \left(\frac{N+1}{2}\right) N \left(1 - \frac{h\alpha}{\theta}\right)^{N-1} \tag{3.1.22}$$

for some random $\alpha \in (0,1)$ and therefore it is bounded by

$$\frac{h^2}{\theta^2}\frac{(N+1)N}{2}. \tag{3.1.23}$$

Therefore since $E[Y(\theta)(N+1)^2] < \infty$,

$$\left| J(\theta) - J(\theta - h) - \frac{h}{\theta} E\left[\sum_{i=1}^{N}(\psi_\emptyset - \psi_i)\right]\right| = ho(h) \tag{3.1.24}$$

■

3.2 Two sided SPA estimate

In this subsection we shall extend a perturbation formula obtained by Brémaud and Vázquez-Abad [3] and which resembles that of Gong [10] (eq. (3.1.21)). However it is different and the manner in which it is obtained is different. The formula obtained are, from an algorithmic point of view, more complex than the LRM or the SPA formulas, but have a better behavior from the point of view of variance. They can be useful in the perturbation analysis of discrete event systems for which only a few real experiments are available and therefore the variance issue is crucial. ¿From a theoretical point of view, they are interesting in that they are obtained by reinterpreting the LRM formulas of subsection 2.1.

The setting is exactly that of subsection 2.1. The following notation will be needed for the statement of the main result of this subsection.

Define for $(i_1, ..., i_{n-p}, j_1, ..., j_p)$ such that $\{i_1, ..., i_{n-p}\} \cap \{j_1, ..., j_p\} = \phi$

$$\Psi^{i_1,...,i_{n-p}}_{j_1,...,j_p} \text{ and } N^{i_1,...,i_{n-p}}_{j_1,...,j_p} \tag{3.2.1}$$

as the functionals Ψ and N computed with X_{i_ℓ} set to 1 for $\ell = 1, ..., n-p$ and X_{j_m} set to 0 for $m = 1, ..., p$.

Theorem 3. Under the conditions A and B of subsection 2.1.

$$\frac{d^n J(\theta)}{d\theta^n} = n! E_\theta \left[\sum_{p=0}^{n}\left\{(-1)^p \sum_{\substack{i_1,...,i_{n-p},j_1,...,j_p \\ i_1<\cdots<i_{n-p},j_1<\cdots<j_p \\ \{i_1,...,i_{n-p}\}\cap\{j_1,...,j_p\}=\phi}} \psi^{i_1,...,i_{n-p}}_{j_1,...,j_p} 1_{N^{i_1,...,i_{n-p}}_{j_1,...,j_p} \le \sup\{i_1,...,i_{n-p},j_1,...,j_p\}}\right\}\right] \tag{3.2.2}$$

Expression (3.2.2) looks rather formidable. However it simplifies in a few cases of interest which we now list:

Special Cases

a. N is a stopping time of $\{T_n, Z_n\}$, for instance $N = n_0$ fixed or $N = \sum_{n\ge 0} 1_{T_n \le t_0}$ for fixed t_0. In this case $N^{i_1,...,i_{n-p}}_{j_1-j_p} \equiv N$. For $n=1$ and $n=2$ we then have

$$\frac{dJ(\theta)}{d\theta} = E_\theta\left[\sum_{0\le i\le N}(\psi^i - \psi_i)\right] \tag{3.2.3}$$

and

$$\frac{d^2 J(\theta)}{d\theta^2} = 2E_\theta \left[\sum_{a \le i < j \le N} \left\{ \psi^{ij} + \psi_{ij} - \psi_j^i - \psi_i^j \right\} \right] \tag{3.2.4}$$

b. If N is the return line of some $\mathcal{F}_n$-adapted process $\{Y_n\}$ in some set G, i.e. if

$$N = \inf\{n \ge 1 | Y_n \in G\} \tag{3.2.5}$$

($= +\infty$ if $Y_n \notin G, \forall n \ge 1$), then it was shown in Brémaud and Vázquez-Abad [3] that formula (3.2.3) holds true.

The proof of Theorem 3 is based on the following observations.

Obversation 1. Recall the notation $\bar{X}_i = 1 - X_i$. We have

$$\binom{M_1}{n-p}\binom{M_2}{p} = \sum_{\substack{i_1,\ldots,i_{n-p},j_1,\ldots,j_p \\ i_1<\ldots<i_{n-p},j_1<\ldots<j_p}} \left\{ \left(\prod_{\ell=1}^{n-p} X_{i_\ell}\right)\left(\prod_{m=1}^{p} \bar{X}_{j_m}\right)\right\} 1_{N \ge \sup\{i_1,\ldots,i_{n-p},j_1,\ldots,j_p\}} \tag{3.2.6}$$

Proof. We might start by observing that the quantity inside the curly brackets of (3.2.6) is automatically zero if $\{i_1, ..., i_{n-p}\} \cap \{j_1, ..., j_p\} \neq \phi$ since we then find a product $X_i\bar{X}_i$ for some i. Next we observe that, conditioned on $N \ge \sup\{i_1, ..., i_{n-p}, j_1, ..., j_p\}$, the first product is non null if and only if $\{i_1, ..., i_{n-p}\}$ is a subset of the collection of the M_1 indices j such that $X_j = 1$, and that there are $\binom{M_1}{n-p}$ such products. A similar statement holds for the second product inside the curly brackets of (3.2.6). The formula then follows since the quantity inside the curly brackets is either 0 or 1. ∎

Obversation 2. For fixed $i_1, ..., i_{n-p}, j_1, ..., j_p$ such that $\{i_1, ..., i_{n-p}\} \cap \{j_1, ..., j_p\} = \phi$ the formula

$$dP_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}} = \left(\prod_{\ell=1}^{n-p} \frac{1}{\theta} X_{i_\ell}\right)\left(\prod_{m=1}^{p} \frac{1}{1-\theta}\bar{X}_{j_m}\right) dP \tag{3.2.7}$$

defines a probability measure $P_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}}$ under which

$$X_n, n \in \mathcal{N}; T_n, n \in \mathcal{N}; Z_n; n \in \mathcal{N} \tag{3.2.8}$$

have the same joint law as under P_θ, except that

$$X_{i_\ell} \equiv 1, \ell = 1, ..., n-p; X_{j_m} \equiv 0, m = 1, ..., p.$$

Proof. In view of the independence assumptions under P_θ, $P_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}}$ differs from P_θ only on $\sigma(X_{i_1}, ..., X_{i_{n-p}}, X_{j_1}, ..., X_{j_p})$, and since

$$\{X_{i_\ell} = 1, \ell = 1, ..., n-p; X_{j_m} = 0, m = 1, ..., p\} \equiv \left\{ \frac{dP_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}}}{dP_\theta} > 0 \right\}$$

the event in the left-hand side has $P_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}}$ probability 1. ∎

Proof of Theorem 3. In view of (2.1.7) and (3.2.6)

$$\frac{d^n J(\theta)}{d\theta^n} = n! E_\theta \left[\sum_{p=0}^{n} (-1)^p \sum_{\substack{i_1,\ldots,i_{n-p},j_1,\ldots,j_p \\ i_1<\ldots<i_{n-p},j_1<\ldots<j_p}} \left(\prod_{\ell=1}^{n-p} X_{i_\ell} \frac{1}{\theta}\right) \left(\prod_{m=1}^{p} \bar{X}_{j_m} \frac{1}{1-\theta}\right) 1_{N \geq \sup\{i_1,\ldots,i_{n-p} j_1,\ldots,j_p\}} \right]$$

that is, using (3.2.7)

$$\frac{d^n J(\theta)}{d\theta^n} = n! \sum_{p=0}^{n} (-1)^p \sum_{\substack{i_1,\ldots,i_{n-p},j_1,\ldots,j_p \\ i_1<\ldots<i_{n-p},j_1<\ldots<j_p}} E_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}} \left[\psi 1_{N \geq \sup\{i_1,\ldots,i_{n-p},j_1,\ldots,j_p\}}\right]$$

Formula (3.2.2) follows since

$$\begin{aligned} & E_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}} \left[\psi 1_{N \geq \sup\{i_1,\ldots,i_{n-p},j_1,\ldots,j_p\}}\right] \\ = \; & E_\theta \left[\psi_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}} 1_{N_{j_1,\ldots,j_p}^{i_1,\ldots,i_{n-p}} > \sup\{i_1,\ldots,i_{n-p},j_1,\ldots,i_p\}}\right]. \end{aligned} \tag{3.2.9}$$

■

Remark Condition B can be relaxed to condition B′ of subsection 2.1. This condition B′ is verified in the routing problem, because of the domination structure (the 1-system dominates all θ-systems for $\theta \in [0,1]$), for all cases of interest.

4 Stationary and Ergodic Estimates

4.1 A stationary model for the routing problem

Let $\{T_n\}, n \in \mathcal{Z}$, be a sequence of random times of $\mathcal{R}$ such that

$$\ldots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \ldots \tag{4.1.1}$$

The time T_n is the time of arrival at the gate of some queueing system of customer number n who brings with him a required service time σ_n. This customer is accepted into the system if and only if some random variable $X_n(\theta)$, $\{0,1\}$-valued, takes the value 1.

The control variable $X_n(\theta)$ is modeled as follows

$$X_n(\theta) = 1_{[0,\theta]}(U_n) \tag{4.1.2}$$

for some $\theta \in [0,1]$ and a sequence $\{U_n\}, n \in \mathcal{Z}$, of iid variables uniformly distributed on $[0,1]$ and independent of the sequences $\{T_n\}, \{\sigma_n\}$. In particular

$$P(X_n(\theta) = 1) = \theta \tag{4.1.3}$$

Define for each $C \in \mathcal{B}(\mathcal{R})$

$$N(C) = \sum_{n \in \mathcal{Z}} 1_C(T_n) \tag{4.1.4}$$

and for $C \in \mathcal{B}(\mathcal{R}), L \in \mathcal{B}(\mathcal{R}_+), E \in \mathcal{B}([0,1])$

$$N(C \times (L \times E)) = \sum_{n \in \mathcal{Z}} 1_C(T_n) 1_L(\sigma_n) 1_E(U_n) \tag{4.1.5}$$

The sequence $\{T_n\}$ marked by the sequence $\{\sigma_n, U_n\}$ is supposed t-stationary in the sense that for all m, all $C_k \subset \mathcal{B}(\mathcal{R}), L_k \in \mathcal{B}(\mathcal{R}_+), E_k \in \mathcal{B}([0,1]), k = 1, ..., m$, the random vector

$$(N((C_1 + \tau) \times (L_1 \times E_1)), ..., N((C_m + \tau) \times (L_m \times E_m))) \tag{4.1.6}$$

is independent of $\tau \in \mathcal{R}$, where $C + \tau \stackrel{\text{def}}{=} \{x + \tau; x \in C\}$. It will be more convenient to use the θ_t-framework to describe the above objects. In this framework, there is a measurable space $(\Omega, \mathcal{F})$ on which is given a family $\theta_t : (\Omega, \mathcal{F}) - (\Omega, \mathcal{F}), t \in \mathcal{R}$, of bijections of Ω such that

$$\begin{cases} \theta_0 = identity & \\ \theta_t \circ \theta_s = \theta_{t+s} & \text{for all } t, s \in \mathcal{R} \\ (t, \omega) \to \theta_t \omega & \text{is } \mathcal{B}(\mathcal{R}) \otimes \mathcal{F}/\mathcal{F} \text{ measurable} \end{cases} \tag{4.1.7}$$

The family $\{\theta_t\}, t \in \mathcal{R}$, is called an abstract shift on $(\Omega, \mathcal{F})$ (see Baccelli and Brémaud [1], p.3).

The random sequence $\{T_n\}, \{\sigma_n\}$ and $\{U_n\}$, and the random variables $N(C), N(C \times (L \times E))$ where $C \in \mathcal{B}(\mathcal{R})$, $L \in \mathcal{B}(\mathcal{R}_+)$, $E \subset \mathcal{B}([0,1])$ are defined as above. It is assumed that for all t, ω and C, L, E as above,

$$N(\theta_t \omega, C \times (L \times E)) = N(\omega, (C + t) \times (L \times E)) \tag{4.1.8}$$

Defining for each sequence $\{V_n\}, n \in \mathcal{Z}$, the process $\{V(t)\}, E \in \mathcal{R}$, by

$$V(t) = \sum_{n \subset \mathcal{Z}} V_n 1_{[T_n, T_{n+1})}(t) \tag{4.1.9}$$

Requirement of (4.1.8) for all C, L, E is equivalent to

$$\begin{cases} N(\theta_t \omega, C) = N(\omega, C + t) \\ \sigma(\omega, t) = \sigma(\theta_t \omega, 0) \\ U(\omega, t) = U(\theta_t \omega, 0) \end{cases} \tag{4.1.10}$$

for all t, ω, C.

In the θ_t-framework, there is defined on $(\Omega, \mathcal{F})$ a probability P such that

$$P \circ \theta_t = P, \forall t \in \mathcal{R} \tag{4.1.11}$$

This guarantees that the random vector (4.1.6) has a distribution independent of τ.

That a θ_t-framework is always available is guaranteed by the Palm theory of point processes, and in particular the inverse construction (see [1], p.13, for instance).

The intensity of N

$$\lambda = E[N([0,1])] \tag{4.1.12}$$

is assumed finite.

Define $X_n(\theta)$ as in (4.1.2) and set

$$\sigma_n(\theta) = \sigma_n X_n(\theta) \tag{4.1.13}$$

The input flow $\{T_n, \sigma_n(\theta)\}, n \in \mathcal{Z}$, and all processes or variables derived from it are said to be relative to the "θ-system".

Assume that

$$(P, \theta_t) \text{ is ergodic.} \tag{4.1.14}$$

Then it can be shown that (see [1], p.30, for instance)

$$E^0[\sigma] \stackrel{\text{def}}{=} \lim_{N \uparrow \infty} \frac{1}{N} \sum_{k=1}^{N} \sigma_k \tag{4.1.15}$$

exists. It is assumed that the 1-system is stable, i.e.

$$\rho \stackrel{\text{def}}{=} \lambda E^0[\sigma] < 1 \tag{4.1.16}$$

If the queueing system has one server working at unit speed and an infinite capacity waiting room, condition (4.1.14) imples, by Loynes' theorem ([16]; see also [1], p.34) that for any $\theta \in [0,1]$, there exists a unique finite workload process $\{W(t,\theta)\}, t \subset \mathcal{R}$, verifying

$$\begin{cases} W(t,\omega,\theta) = W(0,\theta_t\omega,\theta) \\ W(t,\theta) = [W(T_n,\theta) - (t-T_n)]^+, t \in [T_n, T_{n+1}) \\ W(T_n,\theta) = W(T_n^-,\theta) + \sigma_n(\theta) \end{cases} \tag{4.1.17}$$

Moreover, for each $\theta \in [0,1]$, there exits an infinity of positive indices n and an infinity of negative indices n such that $W(T_n^-,\theta) = 0$ ([16]; see [1], p.34 and following).

The "regeneration" times of the θ-system are those times T_n for which $W(T_n^-,\theta) = 0$ and $\sigma_n(\theta) > 0$. They form a sequence $\{R_n(\theta)\}, n \in \mathcal{Z}$, strictly ordered, with the convention

$$R_0(\theta) \leq 0 < R_1(\theta) \tag{4.1.18}$$

In Loynes' theory, the workload process is constructed as follows: First one defines, for any $T < 0$, the random variable $M_T(\theta)$ to be the workload at time 0 of the θ-system assuming that the queue was empty at time T. Then

$$W(0,\theta) = \lim_{T \downarrow -\infty} \uparrow M_T(\theta) \tag{4.1.19}$$

It is obvious from this construction that if $\theta' \leq \theta$, then $M_T(\theta) \leq M_T(\theta')$, and therefore $W(0,\theta') \leq W(0,\theta)$, and therefore for all t,

$$\theta' \leq \theta \Rightarrow W(t,\theta') \leq W(t,\theta) \tag{4.1.20}$$

In this sense, the θ-system dominates the θ'-system when $\theta' \leq \theta$. The 1-system of course dominates all other θ-systems for $\theta \in [0,1]$.

4.2 LRM estimates in the stationary case

Let $\theta \in [0,1]$ be fixed. Define the following two sets of indices n : $\mathcal{N}(\theta)$ consists of all the indices n such that $T_n \in [R_0(\theta), R_1(\theta))$, and $\mathcal{M}(\theta)$ is the set of those indices n in $\mathcal{N}(\theta)$ such that $X_n(\theta) = 1$.

We shall consider for any $\theta' \in [0,\theta]$ a functional $Z(\theta')$ of the form

$$Z(\theta') = g(X_n(\theta'), \sigma_n, T_n;\ n \in \mathcal{N}(\theta), n \leq 0) \tag{4.2.1}$$

For instance $Z(\theta') = W(0, \theta')$ the workload at time 0, or $Z(\theta') = X(0, \theta')$ the number of customers at time 0 under any "reasonable" service discipline such as LIFO, FIFO or any discipline that can be implemented as a function of the current busy cycle of the θ'-system under consideration. It should be noted that the form (4.2.1) is available for all interesting functionals of the θ' system, because of the domination property which is particular to the routing problem.

Conditioned on $\mathcal{N}(\theta)$ and $\{T_n, \sigma_n, n \in \mathcal{N}(\theta)\}$, the distribution of $\{X_n(\theta');\ n \in \mathcal{N}(\theta), n \leq 0\}$ is absolutely continuous with respect to that of $\{X_n(\theta);\ n \in \mathcal{N}(\theta), n \leq 0\}$ and denoting

$$\theta' = \theta - h \tag{4.2.2}$$

the corresponding likelihood ratio is

$$L(\theta, h) = \left(1 - \frac{h}{\theta}\right)^{M_1(0,\theta)} \left(1 + \frac{h}{1-\theta}\right)^{M_2(0,\theta)} \tag{4.2.3}$$

where

$$M_1(0, \theta) = \sum_{\substack{n \in \mathcal{N}(\theta \\ n \leq 0}} X_n(\theta),\ M_2(0, \theta) = \sum_{\substack{n \in \mathcal{N}(\theta) \\ n \leq 0}} (1 - X_n(\theta)) \tag{4.2.4}$$

Therefore

$$E[Z(\theta - h)] = E[g(X_n(\theta), \sigma_n, T_n\ n \in \mathcal{N}(\theta), n \leq 0) L(\theta, h)] \tag{4.2.5}$$

that is

$$E[Z(\theta - h)] = E[Z(\theta) L(\theta, h)] \tag{4.2.6}$$

Theorem 1′ of Section 2.1 adapted to the situation under consideration gives, letting $N(\theta)$ be the cardinal of $\{n| \in \mathcal{N}(\theta), n \leq 0\}$,

Theorem 4. If $|Z(\theta')| \leq Y(\theta)$ for all $\theta' \in [0, \theta]$ where

$$E[Y(\theta) N(\theta)^2] < \infty \tag{4.2.7}$$

then

$$\lim_{h \downarrow 0} \frac{E[Z(\theta) - Z(\theta - h)]}{h} = E[Z(\theta) \ell(0, \theta)] \tag{4.2.8}$$

where

$$\ell(0, \theta) = \frac{M_1(0, \theta)}{\theta} - \frac{M_2(0, \theta)}{1 - \theta}. \tag{4.2.9}$$

Example 1. If $Z(\theta')$ is bounded, for instance $Z(\theta') = e^{iuX(0,\theta')}$, $Z(\theta') = 1_{X(0,\theta')=k}$, condition (4.2.7) is just $E[|N(\theta)|^2] < \infty$. ∎

Example 2. For $Z(\theta') = f(X(0, \theta'))$ where f is an increasing function, the dominated structure of the problem implies that $Z(\theta') \leq Z(\theta)$, and the condition (4.2.7) is just $E[|Z(\theta)| N(\theta)^2] < \infty$. ∎

Define for all $t \in \mathcal{R}$

$$\ell(t, \theta) = \frac{M_1(t, \theta)}{\theta} - \frac{M_2(t, \theta)}{1 - \theta} \tag{4.2.10}$$

where

$$\begin{cases} M_1(t, \theta) = \sum_{n \in \mathcal{N}} X_n(\theta) 1_{[R_-(t,\theta), t)}(T_n) \\ M_2(t, \theta) = \sum_{n \in \mathcal{N}} (1 - X_n(\theta)) 1_{[R_-(t,\theta), t)}(T_r) \end{cases} \tag{4.2.11}$$

and $R_-(t,\theta)$ is the last regeneration point before $(\leq)t$ of the θ-system. Clearly

$$\ell(t,\omega,\theta) = \ell(0,\theta_t\omega,\theta) \tag{4.2.12}$$

and therefore, in view of the ergodic hypothesis, defining

$$\frac{dJ(\theta)}{d\theta} = \lim_{h\downarrow 0} \frac{E[Z(\theta) - Z(\theta - h)]}{h} \tag{4.2.13}$$

we have

$$\frac{dJ(\theta)}{d\theta} = \lim_{t\uparrow\infty} \frac{1}{t}\int_0^t Z(s,\theta)\ell(s,\theta)ds, P - a.s. \tag{4.2.14}$$

where $Z(s,\theta) = Z(\theta)\circ\theta_s$. For instance, if $Z(\theta) = f(X(0,\theta)), Z(s,\theta) = f(X(s,\theta))$.

The process $\{Z(t,\theta)\}$ was assumed compatible with the flow $\{\theta_t\}$, in the sense that $Z(t,\omega,\theta) = Z(0,\theta_t\omega,\theta)$ for all t,ω. This in general implies that the system has been started at $-\infty$. In many cases, at least cases of practical interest, there exists a process $\{Z_0(t,\theta)\}$ that is not compatible with $\{\theta_t\}$ but such that

$$Z(t,\theta) = Z_0(t,\theta) \quad \text{for} \quad t > \tau \tag{4.2.15}$$

where τ is a finite random time. In this case of course

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t Z(s,\theta)\ell(s,\theta)ds = \lim_{t\uparrow\infty} \frac{1}{t}\int_0^t Z_0(s,\theta)\ell(0,\theta)ds \tag{4.2.16}$$

This is precisely the situation of interest to us when

$$Z(t,\theta) = f(W(t,\theta), X(t,\theta)) \tag{4.2.17}$$

for instance. Indeed, in a stable G/G/1 queue, the system starting empty at time 0 couples in finite time with the stationary system (see [1], p.45). One then uses

$$Z_0(t,\theta) = f(W_0(t,\theta), X_0(t,\theta)) \tag{4.2.18}$$

where $\{W_0(t,\theta)\}, t\in\mathcal{R}_+$, and $\{X_0(t,\theta)\}, t\in\mathcal{R}_+$, are respectively the workload process and the congestion process for the θ-system starting empty at time 0.

Regenerative form of the LRM estimate

Let $\{R_n(\theta)\}, n\in\mathcal{Z}$ be the sequence of regeneration times for the θ-system and let P_ℓ^0 be the Palm probability measure associated with P and the sequence $\{R_n(\theta)\}$ (see [1], p.8 and following). The sequence $\{R_n(\theta)\}$ forms a stationary point process, under P, of intensity

$$\lambda_\theta = \lambda P(X_0(\theta) = 1)P(W(0-,\theta) = 0) = \frac{1}{E_\theta^0[R_1(\theta)]} \tag{4.2.19}$$

(see [1], p.14, Eq. (4.2.1)) that is

$$\lambda_\theta = \lambda\theta(1 - \rho\theta) \tag{4.2.20}$$

where we have used the classical formula of Queueing Theory "$\pi(0) = 1 - \rho$". The probabilities P and P_0^θ are linked by the inversion formula (see [1], p.13, Eq. (4.1.2a)).

$$E[g] = \lambda_\theta E_\theta^o\left[\int_0^{R_1(\theta)} g\circ\theta_s ds\right] \tag{4.2.21}$$

for any bounded random variable g.

Moreover, in view of the ergodic assumption, we have (see [1], p27 and following)

$$E_\theta^0\left[\int_0^{R_1(\theta)} g \circ \theta_s ds\right] = \lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{R_n}^{R_n+1}(g\circ\theta_s)ds, P.a.s. \tag{4.2.22}$$

Applying these results to

$$g = f(W(0,\theta), X(0,\theta)) \tag{4.2.23}$$

where f is bounded, we find that

$$\frac{dJ(\theta)}{d\theta} = \lambda\theta(1-\rho\theta)\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{R_n}^{R_{n+1}} f(W(s,\theta), X(s,\theta))\ell(s,\theta)ds, P.a.s. \tag{4.2.24}$$

The variance of the estimate

$$\frac{1}{E_\theta^0[R_1(\theta)]}\int_0^{R_1(\theta)} f(W(s,\theta), X(s,\theta))\ell(s,\theta)ds \tag{4.2.25}$$

is of the order of magnitude of $E_\theta^0[R_1]/\theta(1-\theta)$ as the heuristic argument of subsection 2.2 shows.

4.3 SPA estimates in the stationary case

The situation is now that described in subection 4.1. Let $\theta \in [0,1]$ be fixed.

We consider for any $h \in [0,\theta]$, the random variable $Z(\theta - h)$ of the form

$$Z(\theta - h) = \psi(B_n(h), T_n(\theta), \sigma_n(\theta);\ n \in \mathcal{M}(\theta), n \leq 0) \tag{4.2.1}$$

where $\mathcal{M}(\theta)$ is the set of indices n such that $T_n \subset [R_0(\theta), 0]$ and $X_n(\theta) = 1$, and $B_n(h) \in \{0,1\}$, $B_n(h) = 1$ iff for the corresponding $n \in \mathcal{M}(\theta)$, $X_n(\theta - h) = 0$.

For instance $Z(\theta - h) = W(0, \theta - h)$ the workload at time 0 of the $(\theta - h)$-system.

Adapting Theorem 2 to the present situation, we find

Theorem 4. If $|Z(\theta - h)| \leq Y(\theta)$ for all $n \subset [0,\theta]$, where

$$E[Y(\theta)M(\theta)^2] < \infty \tag{4.2.2}$$

where $M(\theta) = \mathrm{card}(\mathcal{M}(\theta) \cap \{n | n \leq 0\})$

$$\lim_{h\downarrow 0}\frac{E[Z(\theta) - Z(\theta - h)]}{h} = E\left[\frac{1}{\theta}\sum_{\substack{i\in\mathcal{M}(\theta)\\ i\leq 0}}(Z(\theta) + Z_i(\theta))\right] \tag{4.2.3}$$

and $Z_i(\theta)$ is $Z(\theta)$ computed with customer i refused (recall that since $i \in \mathcal{M}(\theta)$, i was accepted in the θ-system). As in subsection 2.4 we find

$$\frac{dJ(\theta)}{d\theta} = \tfrac{1}{\theta}E\left[\textstyle\sum_{\substack{i\in\mathcal{M}(\theta)\\ i\leq 0}}(Z(\theta) - Z_i(\theta))\right] \tag{4.2.4}$$

$$= \tfrac{1}{\theta}\lim_{t\uparrow\infty}\textstyle\int_0^t \sum_{\substack{i\in\mathcal{M}(\theta,0)\\ T_i(\theta)\leq p}}(Z(s,\theta) - Z_i(s,\theta))ds \tag{4.2.5}$$

where $Z(s,\theta) = Z(\theta) \circ \theta s$, $Z_i(s,\theta)$ is $Z(0,\theta)$ computed with customer i removed, and $\mathcal{M}(\theta, s)$ is the set of indices n such that $X_n(\theta) = 1$ and $T_n \in [R_-(s,\theta), s]$.

Figure 1 shows the situation for $Z(\theta) = W(0,\theta)$ and $Z(\theta) = X(0,\theta)$, and therefore $Z(\theta) \circ \theta_2 = W(s,\theta)$ and $Z(\theta) \circ \theta_s = X(s,\theta)$ respectively.

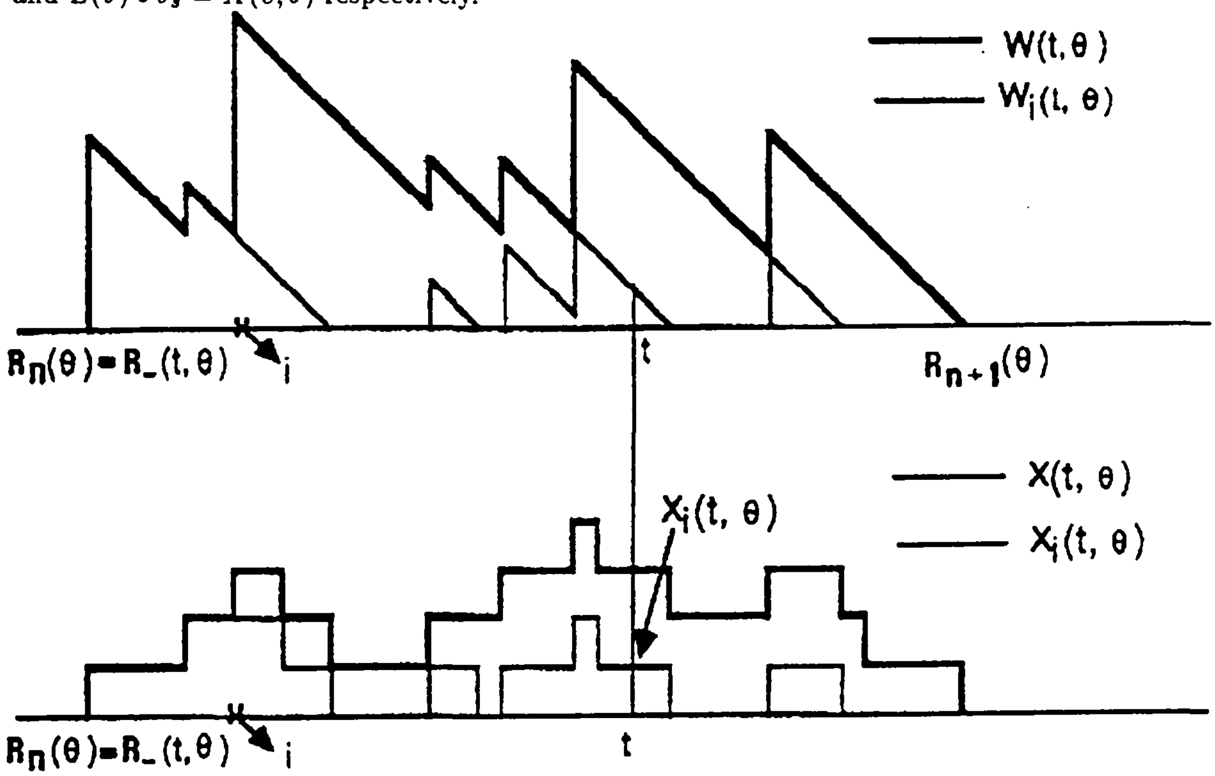

Figure 1. Removing The ith Customer

As in subsection 4.2, we find the regenerative form of the SPA estimates:

$$\frac{dJ(\theta)}{d\theta} = \lambda(1-\rho\theta) \lim \frac{1}{N} \sum_{n=1}^{N} \int_{R_n(\theta)}^{R_{n+1}(\theta)} (Z(s,\theta) - Z_i(s,\theta))ds. \tag{4.2.6}$$

For all $\theta \in [0,1]$ let $Z(\theta)$ be of the form

$$Z(\theta) = (X_n(\theta), T_n,\ Z_n; n \leq 0, n \in \mathcal{M}(1))$$

We then have

Theorem 5. Let $|Z(\theta')| \leq Y$ for all $\theta' \in [0,1]$, and

$$E[YM(1)^2] < \infty \tag{4.2.7}$$

where $M(1) = \text{card } (\mathcal{M}(1) \cap \{n | n \leq 0\})$. then

$$\lim_{h \to 0} \frac{E[Z(\theta+h) - Z(\theta)]}{h} = E\left[\sum_{\substack{i \in \mathcal{M}(1) \\ i \leq 0}} (Z^i(\theta) - Z_i(\theta))\right] \tag{4.2.8}$$

where $Z^i(\theta)$ and $Z_i(\theta)$ are $Z(\theta)$ computed with customer i accepted and customer i refused respectively.

Remark: In this theorem we need to consider the dominating 1-cycle, because adjunction of a customer initially refused in the θ-system might cause an "over flow" of the θ-cycle.

4.4 Application to the adaptive design of routing

Let us consider the following routing problem. There is, somewhere in a network a flow $\{T_n\}, \{\sigma_n\}$, which is to be dispatched to one of two queues, named 1 and 2, with probability θ for queue 1, $1-\theta$ for queue 2. All that is required is that the routing variable does not influence the incoming flow, so that the problem can fit the framework of the present study. This means that feedback is excluded. This restriction can, in theory, be relaxed if we use the GSMP model of Matthes, but we shall not dwell on this (see Glasserman [6] for the use of Matthes models).

Suppose we have a criterion like

$$\min_{\theta} E[X_1(t,\theta) + X_2(t,\theta)] \tag{4.4.1}$$

where $X_i(t,\theta)$ is the number of customers in queue i (waiting line + ticket booths). The problem here is that the value of $X_2(0,\theta)$ cannot be computed over one cycle of the θ-system for queue 1. To deal with this and be able to use the results of Section 2 or 3, we can separate the cost function into

$$J(\theta) = J_1(\theta) + J_2(\theta) \tag{4.4.2}$$

where

$$J_i(\theta) = E[X_i(t,\theta)],\ i = 1, 2 \tag{4.4.3}$$

and compute the estimates of $\frac{dJ_1}{d\theta}(\theta)$ and $\frac{dJ_2(\theta)}{d\theta}$ "separately" on the cycles of queue 1 and 2 respectively. Take for instance the LRM estimate

$$\frac{dJ(\theta)}{d\theta} = E[\ell(0,\theta)X(0,\theta)] \tag{4.4.4}$$

and consider the probability P^0 associated with all the points T_n. The regenerative form of the estimate of $\frac{dJ(\theta)}{d\theta}$ is obtained from the Palm inversion formula

$$E[\ell(0,\theta)X(0,\theta)] = \lambda E^0\left[\int_{T_0=0}^{T_r} \ell(s,\theta)X(s,\theta)ds\right] \tag{4.4.5}$$

Specializing this to queues 1 and 2 we obtain for the estimate of $\frac{dJ(\theta)}{d\theta}$ at iteration k (occurring at T_k)

$$\lambda \int_{T_{k-1}}^{T_k} \{\ell_1(s,\theta)X_1(s,\theta) + \ell_2(s,\theta)X_2(s,\theta)\}\, ds \tag{4.4.6}$$

when

$$\begin{cases} \ell_1(t,\theta) = \frac{M_1^{(1)}(t)}{\theta} - \frac{M_2^{(1)}(t)}{1-\theta} \\ \ell_2(t,\theta) = \frac{M_2^{(2)}(t)}{1-\theta} - \frac{M_1^{(2)}(t)}{\theta} \end{cases} \tag{4.4.7}$$

where $M_1^{(1)}(t)(M_2^{(1)}(t))$ is the number of customer routed to queue 1(2) during the interval $[R_-^1(t), t)$ where $R_-^1(t)$ is the last renewal of queue 1 before time t, and $M_1^{(2)}(t)$ and $M_2^{(2)}(t)$ are the same quantities, only computed over $[R_-^2(t), t)$.

A stochastic gradient algorithm based on this regenerative estimate of $\frac{dJ(\theta)}{d\theta}$ is

$$\theta_{n+1}(\omega) = \theta_n(\omega) - \gamma_{n+1} V_{n+1}(\theta_n(\omega), \omega) \tag{4.4.8}$$

where

$$V_{n+1}(\theta_n) = \lambda \int_{T_n}^{T_{n+1}} \{\ell_1(s, f(\theta_n)) X_1(s, f(\theta_n)) + \ell_2(s, \theta_n) X_2(s, f(\theta_n))\} ds \tag{4.4.9}$$

where

$$f(\theta_n) = \begin{cases} \theta_n & \text{if } \theta_n \in (0,1) \\ 1 & \text{if } \theta_n \geq 1 \\ 0 & \text{if } \theta_n \leq 0 \end{cases}$$

Algorithm (4.4.9) fits the general framework of Métivier and Priouret ([17], Section IV.)

5 Conclusion

In this article we develop stationary gradient estimates for a queueing system with a probabilistic routing mechanism. We obtain likelihood ratio estimate and smoothed perturbation analysis estimate for derivatives of arbitrary order. These estimates can be used for adaptive optimal routing and we include a simple example to illustrate this application.

In contrast to most of the works in the area of simulation-based sensitivity estimation, we don't need the renewal assumption for the arrival and service processes. This is a very desirable feature for network adaptive routing algorithms.

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